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***Problems of Adamjan–Arov–Krein type on subsets of  
the circle and minimal norm extensions***

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# Problems of Adamjan–Arov–Krein type on subsets of the circle and minimal norm extensions

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**Abstract:** We study some generalizations to subsets of the unit circle of Adamjan–Arov–Krein type problems and mainly the one of extending a given function to the missing part of the boundary so as to make it as close to meromorphic with  $N$  poles as possible in the *sup* norm while meeting some gauge constraint. To make our analysis computationally effective, a generic non–multipleness result of the singular values of Hankel operators is established which allows us to provide a convergent resolution algorithm in separable Hölder–Zygmund classes.

**Key-words:** Uniform meromorphic approximation, extremal problems, Adamjan–Arov–Krein extension.

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# Problèmes d'Adamjan–Arov–Krein sur des sous-ensembles du cercle et extensions de norme minimale

**Résumé :** Ce travail concerne la généralisation à des sous-ensembles du cercle unité de problèmes du type Adamjan–Arov–Krein et principalement celui de l'extension d'une fonction donnée à la partie manquante du bord de façon à la rendre la plus proche possible en norme du sup des fonctions méromorphes avec  $N$  pôles sous une certaine contrainte de gabarit. Dans le but de rendre cette analyse numériquement effective, nous établissons un résultat générique de simplicité des valeurs singulières d'opérateurs de Hankel qui nous permet de fournir un algorithme de résolution convergent dans les classes de Hölder–Zygmund séparables.

**Mots-clés :** Approximation méromorphe uniforme, problèmes extremaux, extension d'Adamjan–Arov–Krein.

# 1 Introduction

## 1.1 Framework and motivation

The problem under study in this paper may be considered as a means of making the analytic continuation principle computationally effective in the disk for a function which is merely known on a subset of the circle. This issue arises in a variety of inverse problems ranging from deconvolution and the identification of linear control systems or stochastic processes to inverse scattering and Neumann–Dirichlet problems, see *e.g.* [7, 10, 12, 15, 14, 28, 32, 34] and their bibliography. Although the Carleman theory formally provides a solution as the limit of certain singular integrals [2], the question is well-known to be ill-posed [25]: if, for instance, the boundary data do not *exactly* match those of an analytic function, then Carleman’s recovery formulas diverge [4]. We shall in fact take up a slightly more general issue by allowing the function to be recovered to have a prescribed number of poles, this being a minor extension of the previous question.

We will state this recovery issue as an extremal problem with *a priori* constraints on the portion of the boundary where no data are available. It appears then as a natural generalization to arbitrary subsets of the unit circle  $\mathbb{T}$  of an Adamjan–Arov–Krein (in short, AAK) problem [1]. Providing as it does a link between meromorphic approximation and Hankel operators, the AAK theory has engaged the interest of many researchers in functional analysis [33, 36] and has already found many applications to approximation, interpolation, control theory and signal processing [17, 21, 28, 34].

The present paper can be seen partly as a sequel to [6, 7], where a version of the Nehari problem, based on data provided on a proper subset of the circle, was applied to the question of band-limited frequency-domain identification of stable linear systems. Here we resolve the problem of finding the best completion of a partial model—i.e., of extending a function to the missing part of the boundary so as to make it as close to meromorphic with  $N$  poles as possible in the *sup* norm while meeting some gauge constraint.

In order to keep the solution to this problem constructive, we show that it is generically well behaved in Hölder–Zygmund classes from the numerical viewpoint; this will rest on continuity properties of meromorphic approximation [29, 31] and on the generic non–multiplicity of Hankel singular values. An algorithm is described at the end of the paper.

## 1.2 Notations and preliminaries

In the following,  $\mathbb{D}$  will denote the unit disk of the complex plane and  $\mathbb{T}$  the unit circle. We abbreviate the set of positive real numbers by  $\mathbb{R}^+ := \{x \in \mathbb{R}, x > 0\}$ . When  $E \subset \mathbb{T}$ , we write  $C(E)$  for the space of continuous complex-valued functions on  $E$  while  $L^p(E)$  designates the familiar Lebesgue space for  $1 \leq p \leq \infty$ . We let  $H^p \subset L^p(\mathbb{T})$  be the Hardy space with exponent  $p$  of  $\mathbb{D}$  consisting of functions with vanishing Fourier coefficients of negative index. Alternatively,  $H^p$  is the set of nontangential limits of functions analytic in  $\mathbb{D}$  whose modulus has uniformly bounded  $L^p$ -mean over circles centered at 0 included in  $\mathbb{D}$ ; when  $p = \infty$ , this simply says that the function is bounded in  $\mathbb{D}$ . When  $p = 2$ , we also introduce the conjugate Hardy space  $\bar{H}_0^2$  which is the orthogonal complement to  $H^2$  in  $L^2(\mathbb{T})$ , that is

to say the subspace of functions with vanishing Fourier coefficients of non-negative index. Equivalently,  $f \in \bar{H}_0^2$  if, and only if,  $\bar{f} \in H^2$  and  $\bar{f}$  has zero mean on  $\mathbb{T}$ .

The norm on  $L^p(E)$  will be the natural one, denoted by  $\|\cdot\|_{L^p(E)}$ . In  $L^\infty(E)$ , we write  $d(\phi, S)$  for the distance of the element  $\phi$  to the subset  $S$ ; we use the same notation regardless of  $E \subset \mathbb{T}$ , but the context will keep the meaning clear.

The balls in  $L^\infty(E)$  are weak-\* compact. We use this well-known fact on several occasions in its sequential form: any sequence  $(f_n)$  with  $\|f_n\|_{L^\infty(E)} \leq C$  contains a subsequence  $(f_{n_k})$  such that  $\lim_{k \rightarrow \infty} \int_E f_{n_k} h d\theta = \int_E f h d\theta$  for every  $h \in L^1(E)$  and some  $f$  satisfying  $\|f\|_{L^\infty(E)} \leq C$ . We also recall a theorem of Douglas [20] that the space  $H^\infty + C(\mathbb{T})$  is a subalgebra of  $L^\infty(\mathbb{T})$ . The subscript  $|_E$  applied to a function or to a set of functions indicates restriction to  $E$ ; for instance,  $H_{|E}^p$  is the space of traces on  $E$  of  $H^p$  functions. Whenever  $f$  is defined on  $E$  and  $h$  is defined on its complement  $\mathbb{T} \setminus E$ , then  $f \vee h$  stands for the concatenated function which is defined on all of  $\mathbb{T}$ .

Recall that the modulus of continuity of a function  $f \in C(E)$  is defined by

$$\omega_f(\delta) := \sup_{\substack{x, y \in E \\ |x-y| \leq \delta}} |f(x) - f(y)|.$$

For  $0 < \alpha < 1$  fixed, introduce the Hölder–Zygmund<sup>1</sup> class:

$$\Lambda_\alpha(E) := \{f \in C(E), \omega_f(\delta) = O(\delta^\alpha)\},$$

with norm

$$\|f\|_\alpha := \|f\|_{L^\infty(E)} + \sup_{\delta > 0} \frac{\omega_f(\delta)}{\delta^\alpha}.$$

We define the *separable* Hölder–Zygmund class  $\lambda_\alpha(E)$  to be the subspace of those functions  $f$  in  $\Lambda_\alpha(E)$  such that

$$\frac{\omega_f(\delta)}{\delta^\alpha} \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

We will use the following subsets:

$$\begin{aligned} \lambda_\alpha^+(E) &:= \{f \in \lambda_\alpha(E), f \geq 0\}, \\ \lambda_{\alpha,*}^+(E) &:= \{f \in \lambda_\alpha^+(E), \inf_E f > 0\}, \\ \lambda_{\alpha,0}^+(E) &:= \{f \in \lambda_\alpha^+(E), f = 0 \text{ at boundary points of } E\}. \end{aligned}$$

We also introduce

$$\begin{aligned} \mathcal{M} &:= \{M \in L^\infty(\mathbb{T} \setminus K), M \geq 0 \text{ a.e. on } \mathbb{T} \setminus K\}, \\ \mathcal{M}_* &:= \{M \in \mathcal{M}, \inf_{\mathbb{T} \setminus K} M > 0\}. \end{aligned}$$

The symbol  $\|\cdot\|_{op}$  will be reserved for the norm of an operator from one Banach space into another.

For  $N \geq 0$ , we let  $P_N[z]$  be the space of complex algebraic polynomials in the variable  $z$  of degree at most  $N$ , and we further define  $R_N \subset L^\infty(\mathbb{T})$  to be the set of rational functions with no poles on  $\mathbb{T}$  and at most  $N$  poles in  $\mathbb{D}$ . Finally,  $T$  denotes the space of trigonometric polynomials.

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<sup>1</sup>Also commonly called Lipschitz class.

### 1.3 About AAK approximation

An AAK approximation problem consists by definition in solving for

$$\min_{g \in H^\infty + R_N} \|h - g\|_{L^\infty(\mathbb{T})}, \quad (1)$$

where  $h \in L^\infty(\mathbb{T})$  is given. The following facts are well-known [27, 33, 36]:

- (a) a solution  $g_N$  does exist;
- (b) the solution need not be unique but it is unique at least if  $h \in H^\infty + C(\mathbb{T})$ , and the error function  $|h - g_N|$  is then constant a.e. on  $\mathbb{T}$ .

Very much like for conjugate functions, it turns out that  $g_N$  need not be continuous even if  $f$  is, but  $g_N$  will for instance be continuous if  $f$  is Dini-continuous. The solution to the AAK problem above is closely related to the *Hankel operator with symbol  $h$*

$$\Gamma_h : H^2 \rightarrow \bar{H}_0^2 \quad \text{given by} \quad \Gamma_h v = P_{\bar{H}_0^2}(h v), \quad (2)$$

where  $P_{\bar{H}_0^2}$  is the orthogonal projection from  $L^2(\mathbb{T})$  onto  $\bar{H}_0^2$ . Specifically, a classical theorem of Hartman asserts that  $\Gamma_h$  is compact if and only if  $h \in H^\infty + C(\mathbb{T})$ , and in this case, if  $v_N$  is an eigenvector of  $\Gamma_h^* \Gamma_h$  associated with the  $N + 1$ -st largest eigenvalue, we have that

$$g_N = h - \frac{\Gamma_h v_N}{v_N};$$

moreover  $|h - g_N|$  is equal almost everywhere on  $\mathbb{T}$  to the square root of this eigenvalue<sup>2</sup> which is thus the value of problem (1).

A well-known theorem of Kronecker [27, 36] asserts that  $\Gamma_h$  has finite rank if, and only if,  $h$  belongs to  $H^\infty + R_m$  for some  $m$ . This is the only case where AAK approximants can easily be computed, for then the non-zero singular values and their associated vectors can be obtained as those of the finite matrix representing  $\Gamma_h^* \Gamma_h$  on the orthogonal space to  $\text{Ker } \Gamma_h$  in  $H^2$ . This matrix is easily computed from a rational symbol [17, 18, 21]. Due to the lack of continuity of the best approximation operator  $C(\mathbb{T}) \rightarrow H^\infty + R_N$  [26, 31], situations where no rational symbol is available are much more delicate, but they can be handled generically in smoother classes of functions, see [22, 29] and the last two sections of the present paper.

### 1.4 Statement of the problems

Let  $K$  be a subset of  $\mathbb{T}$ . We refer to the following as the *Bounded Completion Problem*:

**Problem 1** For  $\psi \in L^\infty(\mathbb{T} \setminus K)$  and  $M \in \mathcal{M}$ , define:

$$\mathcal{D}_{M,\psi} := \{h \in L^\infty(\mathbb{T} \setminus K), |h - \psi| \leq M \text{ a.e. on } \mathbb{T} \setminus K\}.$$

Given  $f \in L^\infty(K)$  and  $N \geq 0$  an integer, we seek  $h_N \in \mathcal{D}_{M,\psi}$  such that

$$d(f \vee h_N, H^\infty + R_N) = \min_{h \in \mathcal{D}_{M,\psi}} d(f \vee h, H^\infty + R_N) := \gamma_N(f \vee \psi, M). \quad (3)$$

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<sup>2</sup>This quantity is the  $N + 1$ -st singular value of  $\Gamma_h$  to be defined in a slightly different but equivalent manner in (14).



In words, we are given a bounded function on a subset of the circle and a gauge on the complementary subset, and we seek an extended definition to the whole circle that meets the gauge and makes the global function as close as possible to a meromorphic function with  $N$  poles. Companion to problem 1 will be the *Bounded Extremal Problem*:

**Problem 2** For  $\psi \in L^\infty(\mathbb{T} \setminus K)$ ,  $M' \in \mathcal{M}$ , and  $N \geq 0$  an integer, define:

$$\mathcal{B}_{M',\psi}^N := \{g \in H^\infty + R_N, |g - \psi| \leq M' \text{ a.e. on } \mathbb{T} \setminus K\}.$$

Given  $f \in L^\infty(K)$ , we seek  $g_N \in \mathcal{B}_{M',\psi}^N$  such that

$$\|f - g_N\|_{L^\infty(K)} = \min_{g \in \mathcal{B}_{M',\psi}^N} \|f - g\|_{L^\infty(K)} := \beta_N(f \vee \psi, M'). \quad (4)$$

The bounded extremal problem is close in spirit to the bounded completion: here again we are given a bounded function on a subset of the circle and a gauge on the complementary subset, but this time we directly seek a meromorphic function with  $N$  poles which is closest to the function while meeting the gauge.

When  $f$  and  $\psi$  are fixed and understood from the context, we occasionally write  $\beta_N(M')$  for  $\beta_N(f \vee \psi, M')$  and  $\gamma_N(M)$  for  $\gamma_N(f \vee \psi, M)$ .

If  $K = \mathbb{T}$ , then  $\mathcal{B}_{M',\psi}^N = H^\infty + R_N$  and problem 2 reduces to AAK approximation, while problem 1 is devoid of meaning. On the other hand, if  $K = \emptyset$ , then problem 2 makes no sense and problem 1 amounts to finding a function  $h_N$  whose distance  $\gamma_N$  to  $H^\infty + R_N$  is minimal in  $\mathcal{D}_{M,\psi} \subset L^\infty(\mathbb{T})$ . This non-classical question contains *in nuce* all the ingredients of our approach. Specifically, let  $w_{M+\varepsilon}$  denote for any  $\varepsilon > 0$  the outer factor with modulus  $M+\varepsilon$  and  $\sigma_{N+1}$  be the  $(N+1)$ -st singular value of the Hankel operator with symbol  $\psi w_{M+\varepsilon}^{-1}$ . Then, either  $\sigma_{N+1} \leq 1$  for every  $\varepsilon > 0$ , in which case  $(H^\infty + R_N) \cap \mathcal{D}_{M,\psi} \neq \emptyset$  and  $h_N$  can be any member of the intersection, or else  $\gamma_N$  is implicitly defined as the unique positive real number  $\varepsilon$  such that  $\sigma_{N+1} = 1$ ; in this case, the solution is

$$h_N = \frac{M}{M + \gamma_N} v_N w_{M+\gamma_N} + \left(1 - \frac{M}{M + \gamma_N}\right) \psi,$$

where  $v_N$  is the best AAK approximant with  $N$  poles to  $\psi w_{M+\gamma_N}^{-1}$ . The patient reader will have no problem in verifying this by mimicking the proofs of theorems 3 and 4.

We shall study problem 1 when both  $K$  and  $\mathbb{T} \setminus K$  have positive Lebesgue measure, and we find that it reduces to problem 2 which in turn reduces, although in an implicit manner, to AAK approximation. When  $N = 0$ , problem 2 was analysed by the authors in [6] when  $M'$  is a constant, and in [7] for positive continuous  $M'$ ; we shall carry these results over to  $N \geq 0$  and to arbitrary constraints  $M' \in \mathcal{M}_*$  on our way to solving problem 1. In another connection, we mention that  $L^2$  versions of problems 1 and 2 have been treated in [4, 8] for  $N = 0$  in which case problem 2 was also considered in a Hardy-Sobolev class [3]. But for  $N > 0$ , where the  $L^2$  analog of AAK approximation is best rational approximation in  $H^2$ , the corresponding questions are open; an  $L^p$  version of the AAK problem on  $\mathbb{T}$  is proposed in [9] for  $p \geq 2$  although the occurrence of local minima may hinder numerical computations.

The outline of the paper is as follows. In section 2 we prove the existence of a solution to problem 1 and, when the set  $\mathcal{B}_{M',\psi}^N$  is not empty, to problem 2. In section 3, we show that problem 1 reduces to problem 2 which is itself implicitly equivalent to AAK approximation. In section 4, assuming that  $f \vee \psi \in H^\infty + C(\mathbb{T})$  and  $K$  has an interior point, we establish the continuity of  $\gamma_N$  and  $\beta_N$  along with a reciprocity relation that make our reduction to AAK approximation computationally effective; we then get uniqueness and error circularity for both problems 1 and 2. Section 5 deals with generic non-multipleness of Hankel singular values and contains the technical developments needed to show that problem 1 is well-posed, provided  $(f \vee \psi, M)$  lies in some dense and open subset of  $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$ . The  $\alpha$ -Lipschitz smoothness required from  $M$  and  $f \vee \psi$  is reasonable for many applications. That  $M$  vanishes at the endpoints of  $K$  may seem unnecessarily restrictive at first glance but is really needed if one wants  $f \vee h_N$  to be continuous, a mandatory feature in most modelling problems. Section 6 subsequently describes a constructive algorithm in this case.

*From now on, we tacitly assume that both  $K$  and  $\mathbb{T} \setminus K$  have positive Lebesgue measure.*

## 2 Existence of solutions

Recall that a Blaschke product of degree  $m$  is a rational function of the form

$$b(z) = e^{i\varphi} \prod_{j=1}^m \frac{z - a_j}{1 - \bar{a}_j z},$$

where  $\varphi$  is real and the  $a_j$  are complex numbers of modulus less than 1. Such functions may be characterized as being analytic on the closed disk and having modulus 1 on  $\mathbb{T}$  [20].

**Lemma 1** *For any integer  $N \geq 0$ ,*

$$H^\infty + R_N = \{b^{-1}g : b, g \in H^\infty, b \text{ a Blaschke product of degree } \leq N\} \quad (5)$$

*is weak-\* closed in  $L^\infty(\mathbb{T})$ .*

*Proof:* first, observe that the right-hand side of (5) is indeed equal to  $H^\infty + R_N$  since the latter consists of functions meromorphic in  $\mathbb{D}$  with at most  $N$  poles that remain bounded near  $\mathbb{T}$ . Let  $(\phi_j)$  be a sequence of functions in  $H^\infty + R_N$  which converges weakly-\* to  $\phi \in L^\infty(\mathbb{T})$ . Then, the sequence of Hankel operators  $(\Gamma_{\phi_j})$  converges weakly to  $\Gamma_\phi$ , and it follows from Kronecker's theorem that all of these operators have rank at most  $N$  and hence that  $\phi \in H^\infty + R_N$ . Thus,  $H^\infty + R_N$  is weak-\* sequentially closed, and the lemma follows because  $L^1(\mathbb{T})$  is separable. ■

**Theorem 1** *Problem 1 admits a solution.*

*Proof:* let  $(h_j)$  in  $\mathcal{D}_{M,\psi}$  be a minimizing sequence for (3). By weak-\* compactness,  $(h_j)$  converges weakly-\* to some  $h \in \mathcal{D}_{M,\psi}$  up to a subsequence. By property (a) of AAK approximation in section 1.3, there exists for each  $j$  a function  $k_j \in H^\infty + R_N$  such that

$\|f \vee h_j - k_j\|_{L^\infty(\mathbb{T})} = d(f \vee h_j, H^\infty + R_N)$ . Up to a subsequence,  $(k_j)$  converges weakly-\* to some  $k \in H^\infty + R_N$ , by lemma 1, and

$$\|f \vee h - k\|_{L^\infty(\mathbb{T})} \leq \lim_{j \rightarrow \infty} \|f \vee h_j - k_j\|_{L^\infty(\mathbb{T})} = \gamma_N(f \vee \psi, M),$$

so that  $h$  provides us with a solution. ■

As far as existence is concerned, there is a slight difference between problems 1 and 2, namely the set  $\mathcal{B}_{M', \psi}^N$  of candidate approximants may in the latter case be empty. This happens, for example, when  $d(\psi, (H^\infty + R_N)|_{\mathbb{T} \setminus K}) > \sup_{\mathbb{T} \setminus K} M'$ , which is perfectly possible as there exist functions at positive distance from  $(H^\infty + R_N)|_{\mathbb{T} \setminus K}$  in  $L^\infty(\mathbb{T} \setminus K)$ . Indeed, using lemma 1, the proof of [6, lem.1] is easily adapted to show that any  $\xi \in H^\infty$  verifying

- $\|\xi\|_{L^\infty(\mathbb{T} \setminus K)} \leq 1$ ,
- $\xi|_{\mathbb{T} \setminus K}^{-1} \in L^\infty(\mathbb{T} \setminus K)$ , and
- the zeros of  $\xi$  either accumulate at some interior point of  $\mathbb{T} \setminus K$  or else accumulate nontangentially at some density point of  $\mathbb{T} \setminus K$ ,

is such that  $\xi|_{\mathbb{T} \setminus K}^{-1}$  is at distance at least 1 from  $(H^\infty + R_N)|_{\mathbb{T} \setminus K}$ . In particular,  $\mathcal{B}_{M', \xi^{-1}}^N = \emptyset$  for all  $N \geq 0$  whenever  $M' \in \mathcal{M}$  verifies  $\sup_{\mathbb{T} \setminus K} M' < 1$ . Describing those  $\psi$  for which  $\mathcal{B}_{M', \psi}^N \neq \emptyset$  for all  $M' \in \mathcal{M}_*$  (or even for constant  $M' > 0$ ) is equivalent to characterizing the closure of  $(H^\infty + R_N)|_{\mathbb{T} \setminus K}$  in  $L^\infty(\mathbb{T} \setminus K)$ , and this is an open issue even when  $K$  is an arc. To check that  $\mathcal{B}_{M', \psi}^N \neq \emptyset$ , we shall rely on an elementary sufficient condition:

**Lemma 2** *If  $K \subset \mathbb{T}$  has an interior point and  $\psi \in (H^\infty + C(\mathbb{T}))|_{\mathbb{T} \setminus K}$ , then  $\mathcal{B}_{M', \psi}^N \neq \emptyset$  for every  $M' \in \mathcal{M}_*$  and every integer  $N \geq 0$ .*

*Proof:* it follows immediately from Runge's theorem that already  $\mathcal{B}_{M', \psi}^0 \neq \emptyset$  in this case, see the proof of [6, thm.1,(iii)]. ■

**Remark 1** Lemma 2 shows that problem 1 would not have any solution in general if we omitted the constraint  $h \in \mathcal{D}_{M, \psi}$ . Indeed, if  $\mathbb{T} \setminus K$  has an interior point and  $f \in (H^\infty + C(\mathbb{T}))|_K$ , we can find a sequence of functions in  $H^\infty + R_N$  that will converge to  $f$  on  $K$ , although the argument in [6, prop.2] shows this sequence is necessarily unbounded on  $\mathbb{T} \setminus K$  if  $f \notin (H^\infty + R_N)|_K$ , see also [23, thm.16]. A similar remark holds for problem 2 if the constraint  $g \in \mathcal{B}_{M', \psi}^N$  is omitted.

**Theorem 2** *Problem 2 admits a solution provided  $\mathcal{B}_{M', \psi}^N \neq \emptyset$ .*

*Proof:* let  $(g_j)$  in  $\mathcal{B}_{M', \psi}^N$  be a minimizing sequence for (4). Since it is bounded in  $L^\infty(\mathbb{T})$  some subsequence converges weakly-\* to a function  $g \in H^\infty + R_N$  in view of lemma 1. Up to another subsequence, we may assume that  $M' - |\psi - g_j|$  also converges weakly-\* in  $L^\infty(\mathbb{T} \setminus K)$ . Because weak-\* limit preserves non-negativity, we now see that  $g \in \mathcal{B}_{M', \psi}^N$  and that:

$$\|f - g\|_{L^\infty(K)} \leq \lim_{j \rightarrow \infty} \|f - g_j\|_{L^\infty(K)} = \beta_N(f \vee \psi, M').$$
■

### 3 Reduction to AAK approximation

We first show how the bounded completion problem 1 reduces to a bounded extremal one of type 2. It will be convenient formally to define the trace of  $\mathcal{B}_{M',\psi}^N$  on  $K$ :

$$\mathcal{C}_{M',\psi}^N = \{g|_K, g \in \mathcal{B}_{M',\psi}^N\} \subset L^\infty(K).$$

**Theorem 3** *Let  $f \in L^\infty(K)$ ,  $\psi \in L^\infty(\mathbb{T} \setminus K)$ , and  $N \geq 0$  an integer. For  $M \in \mathcal{M}$ , set*

$$M' = M + \gamma_N(f \vee \psi, M) \text{ on } \mathbb{T} \setminus K. \quad (6)$$

*Then  $\mathcal{B}_{M',\psi}^N \neq \emptyset$  and we have the inequality:*

$$\beta_N(M') \leq \gamma_N(M). \quad (7)$$

*If  $f \notin \mathcal{C}_{M,\psi}^N$ , so that  $\gamma_N(M) > 0$  whence  $M' \in \mathcal{M}_*$ , and if  $g_N$  is a solution to problem 2, then*

$$h_N = \frac{M}{M'} g_N + \left(1 - \frac{M}{M'}\right) \psi \quad (8)$$

*is a solution to problem 1; in fact,*

$$\|f \vee h_N - g_N\|_{L^\infty(\mathbb{T})} = \gamma_N(M).$$

*Proof:* let  $h_N^1 \in \mathcal{D}_{M,\psi}$  be a solution to problem 1 and  $g_N^1 \in H^\infty + R_N$  be such that  $\|f \vee h_N^1 - g_N^1\|_{L^\infty(\mathbb{T})} = \gamma_N(M)$ . By the triangle inequality

$$|\psi - g_N^1| \leq M + \gamma_N(M) = M', \text{ a.e. on } \mathbb{T} \setminus K,$$

so that  $\mathcal{B}_{M',\psi}^N \neq \emptyset$  as it contains  $g_N^1$ . Moreover, by the very definition of  $\beta_N$ , (7) holds. Consider now the function  $h_N$  given by (8). Since

$$|\psi - h_N| = \frac{M}{M'} |\psi - g_N| \leq M, \text{ a.e. on } \mathbb{T} \setminus K,$$

we see that  $h_N \in \mathcal{D}_{M,\psi}$ . Furthermore

$$\begin{aligned} \|f \vee h_N - g_N\|_{L^\infty(\mathbb{T})} &= \max(\|f - g_N\|_{L^\infty(K)}, \|h_N - g_N\|_{L^\infty(\mathbb{T} \setminus K)}) \\ &\leq \max(\beta_N(M'), \|M' - M\|_{L^\infty(\mathbb{T} \setminus K)}) = \gamma_N(M) \end{aligned}$$

where the last equality follows from (6) and (7). ■

**Remark 2** If  $f \in \mathcal{C}_{M,\psi}^N$ , the unique solution to problem 1 is the trace  $h_N$  of  $f$  on  $\mathbb{T} \setminus K$  and  $\gamma_N(M) = 0$  in this case; this function is unambiguously defined because members of  $H^\infty + R_N$  are uniquely determined by the values they assume on a subset of  $\mathbb{T}$  of positive measure.

Next we introduce the AAK approximation to which problem 2 will turn out to be equivalent. For  $\rho \in L^\infty(\mathbb{T})$  such that  $\rho \geq 0$  and  $\log \rho \in L^1(\mathbb{T})$ , let  $w_\rho$  be the outer factor whose value at the origin is positive and whose modulus on  $\mathbb{T}$  is  $\rho$ :

$$w_\rho(z) := \exp \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{it} + z}{e^{it} - z} \log \rho(e^{it}) dt \right\}, \quad (9)$$

Observe that whenever  $\inf_{\mathbb{T}} \rho > 0$ , then  $w_\rho$  is invertible in  $H^\infty$  and  $w_\rho^{-1} = w_{1/\rho}$ .

**Problem 3** *Notation being as in problem 2 with  $M' \in \mathcal{M}_*$ , and assuming  $f \notin \mathcal{C}_{M', \psi}^N$ , we seek  $v_N \in H^\infty + R_N$  such that*

$$\|(f \vee \psi)w_{\beta_N(M') \vee M'}^{-1} - v_N\|_{L^\infty(\mathbb{T})} = \min_{v \in H^\infty + R_N} \|(f \vee \psi)w_{\beta_N(M') \vee M'}^{-1} - v\|_{L^\infty(\mathbb{T})}. \quad (10)$$

The equivalence we are aiming at is the following:

**Theorem 4** *Assume that  $M' \in \mathcal{M}_*$ ,  $\mathcal{B}_{M', \psi}^N \neq \emptyset$  and  $f \notin \mathcal{C}_{M', \psi}^N$ . Any solution  $v_N$  to problem 3 is such that  $g_N = v_N w_{\beta_N(M') \vee M'}$  is a solution to problem 2. Conversely, any solution  $g_N$  to problem 2 gives rise to a solution  $v_N = g_N w_{\beta_N(M') \vee M'}^{-1}$  of problem 3, and the value of that problem is equal to 1.*

*Proof:* since  $\inf_{\mathbb{T}} \beta_N(M') \vee M' > 0$  by assumption, the outer factor  $w_{\beta_N(M') \vee M'}$  is invertible in  $H^\infty$  and, as a consequence of lemma 1, we have that  $v_N$  belongs to  $H^\infty + R_N$  if and only if  $g_N$  does. Recalling the definition of  $\beta_N(M')$ ,  $v_N$  is a solution to problem 3 if and only if

$$\|(f \vee \psi)w_{\beta_N(M') \vee M'}^{-1} - v_N\|_{L^\infty(\mathbb{T})} = \max \left( \frac{\|f - g_N\|_{L^\infty(K)}}{\beta_N(M')}, \left\| \frac{\psi - g_N}{M'} \right\|_{L^\infty(\mathbb{T} \setminus K)} \right) = 1.$$

■

**Remark 3** The case where  $f \in \mathcal{C}_{M', \psi}^N$  is not covered by theorem 4 since it entails  $\beta_N(M') = 0$ . However, problem 2 is then trivial, the unique solution being obviously the  $H^\infty + R_N$  function  $g_N$  such that  $f = g_N$  almost everywhere on  $K$ .

Theorems 3 and 4 reduce problem 1 to the AAK problem 3. However, it is still unclear whether we made any progress towards a constructive solution for at least two reasons. The first is that our reduction to AAK approximation is implicit for we do not know  $\gamma_N(M)$  nor  $\beta_N(M + \gamma_N(M))$ . The second reason is that, in general, AAK problems can be constructively solved only when the function to be approximated is rational whereas

$$(f \vee \psi)w_{\beta_N(M + \gamma_N(M)) \vee (M + \gamma_N(M))}^{-1}$$

has a rather bad singularity at boundary points of  $K$ . The remaining of the paper is essentially devoted to filling these gaps when  $f \vee \psi$  and  $M$  are generic functions in separable Hölder–Zygmund classes and  $K$  has non empty interior.

## 4 Continuity, uniqueness, error circularity in $H^\infty + C(\mathbb{T})$

We assume from now on that:

*$K$  has non-empty interior and  $f \vee \psi \in H^\infty + C(\mathbb{T})$ .*

In particular, it follows from lemma 2 that  $\mathcal{B}_{M',\psi}^N \neq \emptyset$  for all  $M' \in \mathcal{M}_*$ .

We first study the behavior of  $\beta_N$  and  $\gamma_N$  as functions of  $f$ ,  $\psi$ ,  $M'$  or  $M$  and the links between them. This material will be used to ensure uniqueness of a solution to problem 1 in section 4.2, as well as the generic convergence of our algorithm, in sections 5 and 6.

### 4.1 Continuity and reciprocity of the problem values

#### Proposition 1

- (i) *The map  $\beta_N$  is continuous  $(H^\infty + C(\mathbb{T})) \times \mathcal{M}_* \rightarrow [0, \infty)$ .*
- (ii) *The map  $\gamma_N$  is continuous  $(H^\infty + C(\mathbb{T})) \times \mathcal{M} \rightarrow [0, \infty)$ .*
- (iii) *For every  $M \in \mathcal{M}$  such that  $f \notin \mathcal{C}_{M,\psi}^N$ , we have*

$$\beta_N(M + \gamma_N(M)) = \gamma_N(M). \quad (11)$$

*Proof:* in view of theorem 2, let  $g_0$  be a solution to problem 2 and  $b$  a Blaschke product of degree at most  $N$  such that  $bg_0 \in H^\infty$  (see lemma 1). Then,  $(bf \vee b\psi, M')$  belongs to  $(H^\infty + C(\mathbb{T})) \times \mathcal{M}_*$  and, since  $\beta_0$  is continuous on  $(H^\infty + C(\mathbb{T})) \times \mathcal{M}_*$  because it is convex and locally bounded (see *e.g.* [13, thm.1.10]), it is readily seen that

$$\beta_N(f \vee \psi, M') = \beta_0(bf \vee b\psi, M') = \lim_{m \rightarrow \infty} \beta_0(bf_m \vee b\psi_m, M'_m), \quad (12)$$

for all sequences  $(f_m \vee \psi_m)$  and  $(M'_m)$  converging uniformly to  $f \vee \psi$  and  $M'$  in  $(H^\infty + C(\mathbb{T}))$  and  $\mathcal{M}_*$ , respectively. Pick  $h_m \in \mathcal{B}_{M'_m, b\psi_m}^0$  such that

$$\beta_0(bf_m \vee b\psi_m, M'_m) = \|bf_m - h_m\|_{L^\infty(K)}.$$

By the definition of  $\beta_N$ , we have

$$\beta_N(f_m \vee \psi_m, M'_m) \leq \|f_m - h_m/b\|_{L^\infty(K)} = \beta_0(bf_m \vee b\psi_m, M'_m),$$

which implies in view of (12) that

$$\limsup_m \beta_N(f_m \vee \psi_m, M'_m) \leq \beta_N(f \vee \psi, M'). \quad (13)$$

This proves the upper semi-continuity of  $\beta_N$ .

Denoting now by  $g_m$  a solution to problem 2 for  $f_m$ ,  $\psi_m$ , and  $M'_m$ , we get:

$$\|g_m\|_{L^\infty(K)} \leq \beta_N(f_m \vee \psi_m, M'_m) + \|f_m\|_{L^\infty(K)}, \quad \|g_m\|_{L^\infty(\mathbb{T} \setminus K)} \leq M'_m + \|\psi_m\|_{L^\infty(\mathbb{T} \setminus K)},$$

so that  $\|g_m\|_{L^\infty(\mathbb{T})}$  is bounded, in view of (13). Some weak-\* limit point  $g \in \mathcal{B}_{M',\psi}^N$  of  $\{g_m\}$  then satisfies

$$\|f - g\|_{L^\infty(K)} \leq \liminf_m \|f - g_m\|_{L^\infty(K)} = \liminf_m \beta_N(f_m \vee \psi_m, M'_m)$$

whence *a fortiori*

$$\beta_N(f \vee \psi, M') \leq \liminf_m \beta_N(f_m \vee \psi_m, M'_m).$$

This proves lower semi-continuity, hence (i) holds. The proof of (ii) is similar and we omit it.

To get (iii), observe that the left hand side of (11) does not exceed the right by theorem 3. To prove equality, pick  $M \in \mathcal{M}$ , set  $M' = M + \gamma_N(M) \in \mathcal{M}_*$  and assume that  $\beta_N(M') < \gamma_N(M)$ . Because we just proved that  $\beta_N$  is continuous, we can find  $\varepsilon > 0$  such that  $M'_1 = M' - \varepsilon \in \mathcal{M}_*$  and  $\beta_N(M'_1) < \gamma_N(M)$ . Let  $g_N^{(1)} \in H^\infty + R_N$  be a solution to problem 2 where  $M'_1$  is substituted for  $M'$ , and set

$$h_N^{(1)} = \frac{M}{M'_1} g_N^{(1)} + \left(1 - \frac{M}{M'_1}\right) \psi.$$

It is easily checked that  $h_N^{(1)} \in \mathcal{D}_{M,\psi}$  and

$$d(f \vee h_N^{(1)}, H^\infty + R_N) \leq \max(\beta_N(M'_1), \gamma_N(M) - \varepsilon) < \gamma_N(M),$$

a contradiction which proves (11). ■

**Corollary 1** *If  $f \notin \mathcal{C}_{M,\psi}^N$ , then  $f \notin \mathcal{C}_{M',\psi}^N$ , with  $M' = M + \gamma_N(M)$ .*

*Proof:* whenever  $f \in \mathcal{C}_{M',\psi}^N \setminus \mathcal{C}_{M,\psi}^N$ , it holds by definition that  $\beta_N(M') = 0 < \gamma_N(M)$  thereby contradicting (11). ■

For  $k \geq 1$ , we denote by  $\sigma_k(\Gamma_\Phi)$  the  $k$ -th singular value of the Hankel operator  $\Gamma_\Phi$ :

$$\sigma_k(\Gamma_\Phi) := \inf_S \{ \|\Gamma_\Phi - S\|_{op}, \text{rank } S < k \}. \quad (14)$$

When  $\Phi \in H^\infty + C(\mathbb{T})$ , then  $\Gamma_\Phi$  is compact and the singular values of  $\Gamma_\Phi$  are the positive square roots of the eigenvalues of  $\Gamma_\Phi^* \Gamma_\Phi$ . We shall have an occasion to use the following lemma.

**Lemma 3** *For  $M \in \mathcal{M}$ , the map*

$$(G, \gamma) \mapsto \sigma_k(\Gamma_{G w_{\gamma \vee (M + \gamma)}^{-1}})$$

*is continuous  $(H^\infty + C(\mathbb{T})) \times (0, \infty) \rightarrow (0, \infty)$  and decreases with  $\gamma$  for fixed  $G$ .*

*Proof:* if  $S_1$  and  $S_2$  are any operators, we have the elementary inequality

$$|\sigma_k(S_1) - \sigma_k(S_2)| \leq \|S_1 - S_2\|_{op}. \quad (15)$$

Therefore, if we pick  $G_1, G_2$  in  $H^\infty + C(\mathbb{T})$  and  $\gamma_1, \gamma_2$  positive, we can write

$$\left| \sigma_k \left( \Gamma_{G_1 w_{\gamma_1 \vee (M + \gamma_1)}^{-1}} \right) - \sigma_k \left( \Gamma_{G_2 w_{\gamma_2 \vee (M + \gamma_2)}^{-1}} \right) \right|$$

$$\leq \left\| \Gamma_{(G_2 - G_1) w_{\gamma_2 \vee (M + \gamma_2)}^{-1}} \right\|_{op} + \left| \sigma_k \left( \Gamma_{G_1 w_{\gamma_2 \vee (M + \gamma_2)}^{-1}} \right) - \sigma_k \left( \Gamma_{G_1 w_{\gamma_1 \vee (M + \gamma_1)}^{-1}} \right) \right|.$$

If we let  $G_2$  and  $\gamma_2$  tend respectively to  $G_1$  and  $\gamma_1$  that are kept fixed, the first term above becomes arbitrarily small since  $\|w_{\gamma_2 \vee (M + \gamma_2)}^{-1}\|_{L^\infty(\mathbb{T})}$  remains bounded, so it remains to show continuity and monotonicity with respect to  $\gamma$  for fixed  $G$ .

For this, note that if  $\Phi \in L^\infty(\mathbb{T})$  and  $\Psi \in H^\infty$ , then  $\Gamma_{\Phi\Psi} = \Gamma_\Phi M_\Psi$ , where  $M_\Psi$  is the operator of multiplication by  $\Psi$ , and hence, by [27, cor.1.5],

$$\sigma_k(\Gamma_{\Phi\Psi}) \leq \sigma_k(\Gamma_\Phi) \|\Psi\|_{L^\infty(\mathbb{T})}. \quad (16)$$

Likewise, if  $\Psi^{-1} \in H^\infty$ , then

$$\sigma_k(\Gamma_{\Phi\Psi}) \|\Psi^{-1}\|_{L^\infty(\mathbb{T})} \geq \sigma_k(\Gamma_\Phi).$$

Consequently, from

$$\Gamma_{G w_{\gamma_2 \vee (M + \gamma_2)}^{-1}} = \Gamma_{G w_{\gamma_1 \vee (M + \gamma_1)}^{-1}} M_{w_{\frac{\gamma_1 \vee (M + \gamma_1)}{\gamma_2 \vee (M + \gamma_2)}}},$$

we deduce for  $0 < \gamma_1 < \gamma_2 < \infty$  that

$$A \sigma_k(\Gamma_{G w_{\gamma_1 \vee (M + \gamma_1)}^{-1}}) < \sigma_k(\Gamma_{G w_{\gamma_2 \vee (M + \gamma_2)}^{-1}}) < \sigma_k(\Gamma_{G w_{\gamma_1 \vee (M + \gamma_1)}^{-1}}), \quad (17)$$

where

$$A = \left\| \frac{\gamma_2 \vee (M + \gamma_2)}{\gamma_1 \vee (M + \gamma_1)} \right\|_{L^\infty(\mathbb{T})}^{-1}.$$

As  $A \rightarrow 1$  when  $\gamma_2 - \gamma_1 \rightarrow 0$ , this shows at the same time the continuity and the monotonicity of the partial map for fixed  $G$ . ■

We turn to an intrinsic characterization of  $\gamma_N$ .

**Proposition 2** *If  $f \notin \mathcal{C}_{M,\psi}^N$ , then  $\gamma_N(M)$  is the unique positive real number  $\gamma$  such that*

$$\min_{v \in H^\infty + R_N} \|(f \vee \psi) w_{\gamma \vee (M + \gamma)}^{-1} - v\|_{L^\infty(\mathbb{T})} = 1. \quad (18)$$

*Proof:* from point (iii) of proposition 1, we get  $\beta_N(M') = \gamma_N(M)$  for  $M' = M + \gamma_N(M)$ . Substituting these quantities in the second assertion of theorem 4 proves that (18) holds for  $\gamma = \gamma_N(M)$ . Since, by AAK theory:

$$\min_{v \in H^\infty + R_N} \|(f \vee \psi) w_{\gamma \vee (M + \gamma)}^{-1} - v\|_{L^\infty(\mathbb{T})} = \sigma_{N+1}(\Gamma_{(f \vee \psi) w_{\gamma \vee (M + \gamma)}^{-1}}),$$

it follows from (17) that the solution to (18) is unique. ■

**Remark 4** Equivalently, if  $f \notin \mathcal{C}_{M',\psi}^N$ , then  $\beta_N(M')$  is the unique positive real number  $\beta$  such that

$$\min_{v \in H^\infty + R_N} \|(f \vee \psi) w_{\beta \vee M'}^{-1} - v\|_{L^\infty(\mathbb{T})} = 1. \quad (19)$$



## 4.2 Uniqueness and error circularity when $f \vee \psi \in H^\infty + C(\mathbb{T})$

We now translate to problems 1 and 2 property (b) of AAK approximation stated in section 1.3. We begin with the bounded extremal problem:

**Theorem 5** *For  $M' \in \mathcal{M}_*$ , the solution  $g_N$  to problem 2 is unique. Whenever  $f \notin \mathcal{C}_{M',\psi}^N$ , it satisfies*

$$\begin{cases} |f - g_N| &= \beta_N(M') > 0 & \text{a.e. on } K, \\ |\psi - g_N| &= M' & \text{a.e. on } \mathbb{T} \setminus K. \end{cases} \quad (20)$$

*Proof:* if  $f \in \mathcal{C}_{M',\psi}^N$ , then uniqueness is automatic as pointed out in remark 3. Assuming now that  $f \notin \mathcal{C}_{M',\psi}^N$ , uniqueness follows from theorem 4 and property (b) of AAK approximation. We also know that the error is circular in this case, so that, in view of (19),  $|(f \vee \psi) w_{\beta_N(M') \vee M'}^{-1} - v_N| = 1$ . Substituting  $g_N w_{\beta_N(M') \vee M'}^{-1}$  for  $v_N$  as indicated in theorem 4 and taking into account that  $w_{\beta_N(M') \vee M'}^{-1}$  has modulus  $1/\beta_N(M')$  on  $K$  and  $1/M'$  on  $\mathbb{T} \setminus K$  leads us to (20). ■

As for the bounded completion, we get:

**Theorem 6** *The solution  $h_N$  to problem 1 is unique. Whenever  $f \notin \mathcal{C}_{M,\psi}^N$ , it satisfies*

$$|\psi - h_N| = M \text{ a.e. on } \mathbb{T} \setminus K. \quad (21)$$

*Proof:* if  $f \in \mathcal{C}_{M,\psi}^N$ , uniqueness clearly holds, see remark 2. Assume now that  $f \notin \mathcal{C}_{M,\psi}^N$ . Let  $h_N^1$  be a solution to problem 1 and  $g_N \in H^\infty + R_N$ , be such that

$$\|f \vee h_N^1 - g_N\|_{L^\infty(\mathbb{T})} = \gamma_N(M). \quad (22)$$

Clearly  $g_N \in \mathcal{B}_{M',\psi}^N$  with  $M' = M + \gamma_N(M)$  and  $\|f - g_N\|_{L^\infty(K)} \leq \gamma_N(M) = \beta_N(M')$ , since (11) holds good from proposition 1. Therefore  $g_N$  solves for problem 2 and, according to theorem 5, is uniquely defined by this property. Moreover, we deduce from theorem 3 that  $h_N$  given by (8) is also a solution to problem 1. A direct computation yields

$$|\psi - h_N| = \frac{M}{M'} |\psi - g_N|,$$

and (20) implies (21). It remains to show that  $h_N^1 = h_N$ . Using (20) again and the definition of  $h_N^1$ , we can write

$$M' = |\psi - g_N| \leq |\psi - h_N^1| + |h_N^1 - g_N| \leq M + \gamma_N = M' \quad \text{a.e. on } \mathbb{T} \setminus K$$

so that equality holds throughout, hence we see for almost every  $e^{i\theta} \in \mathbb{T} \setminus K$  that  $h_N^1(e^{i\theta})$  lies on the segment  $[\psi(e^{i\theta}), g_N(e^{i\theta})]$  at a distance  $M(e^{i\theta})$  from  $\psi(e^{i\theta})$ . Since  $h_N$  also has this property, we get  $h_N^1 = h_N$  as desired. ■

## 5 Generic well-posedness in separable Hölder–Zygmund classes

Before we can describe a constructive solution to the bounded completion problem, at least for generic  $\lambda_\alpha$ -data, we have to address a general difficulty with AAK approximation, namely the map sending a function to its best approximant is discontinuous  $C(\mathbb{T}) \rightarrow H^\infty + R_N$  at every non-meromorphic argument [26]; in particular, uniformly approximating a continuous function by a rational one does *not* guarantee that the best AAK approximation of the rational function will be uniformly close to that of the original function. The situation improves if one restricts to smoother classes of functions. An attractive class here is the separable Hölder–Zygmund space  $\lambda_\alpha(\mathbb{T})$ . Indeed, rational approximants can be obtained in  $\lambda_\alpha(\mathbb{T})$  upon convolving with approximate identities (for example the Fejér kernel, see lemma 4 below); moreover, AAK approximation maps  $\lambda_\alpha(\mathbb{T})$  into itself [31] and is continuous there precisely at those points  $\Phi$  whose associated Hankel operator  $\Gamma_\Phi$  has non-multiple  $N + 1$ -st singular value [29, thm.6]. To warrant this approach to AAK approximation, it only remains to prove the intuitively obvious fact that the multiplicity condition is satisfied most of the time. Therefore, we begin our constructive analysis by establishing that Hankel singular values are generically simple for rather general spaces of continuous functions. In addition to  $\lambda_\alpha(\mathbb{T})$ , these will comprise all hereditary classes of type 2, in the terminology of [30]<sup>3</sup>, where the non-multiplicity of the  $(N + 1)$ -st singular value is known to be necessary (and in most cases sufficient) for the continuity of best meromorphic approximation with  $N$  poles.

**Theorem 7** *Let  $N \geq 1$  be an integer and let  $\mathcal{B}$  be either:*

*(i) the set of complex trigonometric polynomials of degree at most  $L \geq N$ ,*

*or*

*(ii) a complex Banach space of continuous functions on  $\mathbb{T}$  densely containing  $T$ , and such that the natural injection  $\mathcal{B} \rightarrow C(\mathbb{T})$  is continuous.*

*Define*

$$\mathcal{O}_N = \{\Phi \in \mathcal{B}; \sigma_k(\Gamma_\Phi) \text{ has multiplicity } 1 \text{ for } 1 \leq k \leq N\}.$$

*Then  $\mathcal{O}_N$  is open and dense in  $\mathcal{B}$ .*

Let us point out the following corollary, which may be of independent interest and is logically weaker than theorem 7 in case (i) and immediately follows in case (ii) from the Baire category theorem (see *e.g.* [35]):

**Corollary 2** *The set of functions whose associated Hankel operator has simple singular values is dense in  $\mathcal{B}$ .*

*Proof of theorem 7:* it follows from the already used inequality (15) that

$$|\sigma_k(\Gamma_{\Phi+\Delta\Phi}) - \sigma_k(\Gamma_\Phi)| \leq \|\Delta\Phi\|_{L^\infty(\mathbb{T})} \leq C\|\Delta\Phi\|_{\mathcal{B}}, \quad (23)$$

---

<sup>3</sup>For example Besov classes  $B_{p,p}^s$  for  $1 \leq p < \infty$ ,  $s > 1/p$ , as well as  $B_1^1$  or the Wiener algebra  $\mathcal{W}$ .

where  $C$  is a positive constant provided by the assumptions and  $\|\cdot\|_{\mathcal{B}}$  denotes the norm on  $\mathcal{B}$ . If the first  $N$  singular values of  $\Gamma_{\Phi}$  are simple and if we set

$$\min_{1 \leq j \leq N} \sigma_j(\Gamma_{\Phi}) - \sigma_{j+1}(\Gamma_{\Phi}) = \epsilon > 0,$$

it is readily seen from (23) that the first  $N$  singular values of  $\Gamma_{\Phi+\Delta\Phi}$  will remain simple provided  $\|\Delta\Phi\|_{\mathcal{B}} < \epsilon/2C$ . This proves the openness of  $\mathcal{O}_N$ .

It remains to show that  $\mathcal{O}_N$  is dense in  $\mathcal{B}$ . Since in case (ii) the set  $T$  is dense in  $\mathcal{B}$ , it is enough to treat case (i), so we set  $\mathcal{B}$  to be the space of trigonometric polynomials of degree at most  $L$ , where  $L \geq N$ . Since  $\mathcal{B}$  is finite dimensional, the choice of the norm is irrelevant here; to fix ideas, we shall identify  $\mathcal{B}$  with  $\mathbb{C}^{2L+1}$  equipped with the Euclidean topology. We shall prove that trigonometric polynomials whose associated Hankel operator has simple first  $L$  singular values form a dense subset of  $\mathcal{B}$ ; this will *a fortiori* imply what we want.

Notice that the Hankel operator associated to the trigonometric polynomial

$$\sum_{k=-L}^L q_k z^{-k} \in \mathcal{B}, \quad z = e^{i\theta},$$

is equal to  $\Gamma_q$  where  $q$  is the anti-analytic projection

$$q(z) = q_1 z^{-1} + \cdots + q_L z^{-L} \in z^{-1} P_{L-1}[1/z].$$

We single out the subset  $\mathcal{M}_L$  of those  $q \in z^{-1} P_{L-1}[1/z]$  for which  $q_L \neq 0$ . It is clear that  $\mathcal{M}_L$  is open and dense in  $z^{-1} P_{L-1}[1/z]$  when the latter is identified with  $\mathbb{C}^L$ , and it is enough to show that the first  $L$  singular values of  $\Gamma_q$  are simple for  $q$  in a dense subset of  $\mathcal{M}_L$ .

The first  $L$  singular values of  $\Gamma_q$  are the singular values of the finite Hankel matrix

$$\mathcal{H}_q = \begin{pmatrix} q_1 & q_2 & \cdots & q_{L-1} & q_L \\ q_2 & q_3 & \cdots & q_L & 0 \\ \vdots & \vdots & . & 0 & \vdots \\ \vdots & q_L & 0 & \vdots & \vdots \\ q_L & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (24)$$

which is obviously nonsingular as  $q_L \neq 0$  by assumption; the remaining singular values of  $\Gamma_q$  are of course zero.

For  $1 \leq j \leq L$  we write  $q_j = r_j + is_j$ , where  $r_j$  and  $s_j$  are real. Then the singular values of  $\mathcal{H}_q$  are the positive roots of  $\det(\mathcal{H}_q^* \mathcal{H}_q - \sigma^2 I) = 0$ , which is a polynomial equation in  $\sigma$  whose coefficients are themselves polynomials in the variables  $r_j$  and  $s_j$ .

Now a polynomial has no repeated roots if and only if its discriminant  $\Delta$  is nonzero (see for example [11, 19]). As  $\Delta$  is a polynomial in the variables  $r_1, \dots, r_L, s_1, \dots, s_L$ , we have to prove that it is not the zero polynomial, because then the set of coefficients  $q_1, \dots, q_L$  for which it does not vanish will be dense in  $\mathbb{C}^L$  and we shall be done. Now, if  $\Delta$  was identically zero, that would mean that all  $L \times L$  Hankel matrices  $\mathcal{H}_q$  defined by (24) have repeated singular values. However we can easily construct one that does not by induction on  $L$ . For  $L = 1$  the result is clear; for the induction step, supposing that we have a  $L \times L$  Hankel

matrix with top row  $q_1, \dots, q_L$  whose singular values are simple and nonzero, we choose  $q_{L+1}$  very small but non-zero. It is plain if  $|q_{L+1}|$  is sufficiently small, using the continuity of the roots with respect to the coefficients, that the first  $N$  singular values of the  $(L+1) \times (L+1)$  Hankel matrix with parameters  $q_1, \dots, q_{L+1}$  stay close to those of the  $L \times L$  matrix with parameters  $q_1, \dots, q_L$  and therefore remain distinct, while the  $(L+1)$ -st is smaller than the previous ones but non-zero. ■

For many applications of AAK approximation, in particular to engineering and system theory, one deals with functions having *real* Fourier coefficients and theorem 7 tells nothing in this case because such functions already form a small subset of  $\mathcal{B}$ . To warrant such applications, we observe that both theorem 7 and corollary 2 remain true when  $\mathcal{B}$  is either the set of trigonometric polynomials with real coefficients of degree at most  $L \geq N$  or a real Banach space of continuous functions on  $\mathbb{T}$  densely containing the trigonometric polynomials with real coefficients. The proof of this would be virtually identical to the one of theorem 7, the only difference being that coefficients are real throughout.

In order to apply theorem 7 to  $\mathcal{B} = \lambda_\alpha(\mathbb{T})$  and to the symbol

$$(f \vee \psi) w_{\gamma_N(M) \vee (M + \gamma_N(M))}^{-1},$$

we need a couple of lemmas. Though the first one is apparently well-known, we could not locate it in the literature.

**Lemma 4** *If  $\overline{T}^\alpha$  denotes the closure of  $T$  in  $\Lambda_\alpha(\mathbb{T})$ , then  $\overline{T}^\alpha = \lambda_\alpha(\mathbb{T})$ , the separable Hölder–Zygmund class.*

*Proof:* clearly  $T \subset \lambda_\alpha$ , since functions in  $T$  are differentiable. If  $f \in \overline{T}^\alpha$  and  $\epsilon > 0$ , pick  $g \in T$  such that  $\|f - g\|_\alpha < \epsilon/2$ . Since  $\omega_g(\delta) < \epsilon\delta^\alpha/2$  for  $\delta$  small enough, we get for such  $\delta$  and  $0 < \eta \leq \delta$ :

$$\begin{aligned} |f(\theta + \eta) - f(\theta)| &\leq |g(\theta + \eta) - g(\theta)| + |(f - g)(\theta + \eta) - (f - g)(\theta)| \\ &< \epsilon\delta^\alpha/2 + \epsilon\delta^\alpha/2 = \epsilon\delta^\alpha \end{aligned}$$

for every  $\theta \in (-\pi, \pi]$ . Hence  $\omega_f(\delta) = o(\delta^\alpha)$  so that  $f \in \lambda_\alpha(\mathbb{T})$ .

We now show that for  $f \in \lambda_\alpha(\mathbb{T})$ , we have  $\|f_n - f\|_\alpha \rightarrow 0$ , where  $f_n$  is a sequence of trigonometric polynomials obtained by convolving  $f$  with an approximate identity  $K_n$  that may be for instance the Fejér kernel, see *e.g.* [24, 37]. By the well-known properties of such a kernel,  $f_n$  converges uniformly to  $f$ , so it is enough to show that

$$\sup_{\delta > 0} \omega_{f-f_n}(\delta)/\delta^\alpha \rightarrow 0.$$

Let  $\epsilon > 0$  be given, and write  $\omega_f(\delta) = k_\delta\delta^\alpha$ , where  $k_\delta$  is bounded and  $k_\delta \rightarrow 0$  as  $\delta \rightarrow 0$ . Thus  $k_\delta < \epsilon/4$  for  $\delta \leq \delta_1$ , say. Put for simplicity  $(S_\eta f)(\theta) = f(\theta + \eta)$ . We have by definition:

$$\|S_\eta f - f\|_{L^\infty(\mathbb{T})} \leq k_\delta\delta^\alpha, \text{ for } 0 < \eta \leq \delta. \quad (25)$$

Since convolving with  $K_n$  commutes with  $S_\eta$  and does not increase the  $\sup$  norm, we get  $\|S_\eta f_n - f_n\|_{L^\infty(\mathbb{T})} \leq k_\delta \delta^\alpha$ . Hence for  $\delta \leq \delta_1$ :

$$\|S_\eta(f - f_n) - (f - f_n)\|_{L^\infty(\mathbb{T})} < 2k_\delta \delta^\alpha < (\epsilon/2)\delta^\alpha.$$

To get a similar estimate for  $\delta > \delta_1$ , take  $0 < \eta \leq \delta$  and use  $\int_{-\pi}^\pi K_n = 1$  to write:

$$S_\eta(f - f_n)(\theta) - (f - f_n)(\theta) = \int_{-\pi}^\pi [f(\theta + \eta) - f(\theta + \eta - y) - f(\theta) + f(\theta - y)] K_n(y) dy.$$

Split the integral into 2 pieces:  $I_1$  taken over  $|y| \leq \delta_1$ , and  $I_2$  the remainder. By (25) and our choice of  $\delta_1$ , we get  $I_1 < (\epsilon/2)\delta^\alpha$ . As to  $I_2$ , we get the obvious upper bound:

$$I_2 \leq 2k_\delta \delta^\alpha \int_{|y| > \delta_1} K_n(y) dy$$

and  $\int_{|y| > \delta_1} K_n$  becomes smaller than  $\epsilon/4 \sup_\delta k_\delta$  for  $n$  sufficiently large depending only on  $\delta_1$ . Adding all this up we get  $\sup_{\delta > 0} \omega_{f-f_n}(\delta)/\delta^\alpha < \epsilon$  for  $n$  large, as desired.  $\blacksquare$

**Lemma 5** *The function  $\Omega(F, \rho) = F w_\rho$  is continuous  $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,*}(\mathbb{T}) \rightarrow \lambda_\alpha(\mathbb{T})$ . Moreover, for fixed  $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$ , the partial map  $\Omega_\rho(F) = \Omega(F, \rho)$  is a homeomorphism on  $\lambda_\alpha(\mathbb{T})$ .*

*Proof:* since  $\lambda_\alpha(\mathbb{T})$  is a Banach algebra, as follows immediately from:

$$\omega_{ab} \leq \|b\|_{L^\infty(\mathbb{T})} \omega_a + \|a\|_{L^\infty(\mathbb{T})} \omega_b,$$

the continuity of  $\Omega$  reduces to show that  $\rho \mapsto w_\rho$  is continuous  $\lambda_\alpha^{+,*}(\mathbb{T}) \rightarrow \lambda_\alpha(\mathbb{T})$ . To this effect, observe that on  $\mathbb{T}$ :

$$w_\rho = \exp(\log \rho + i \widetilde{\log \rho}).$$

where  $\widetilde{f}$  denotes, for any  $f \in C(\mathbb{T})$ , the boundary value taken by the harmonic conjugate of its Poisson integral. That Hölder–Zygmund classes  $\Lambda_\alpha(\mathbb{T})$  are preserved by the conjugation operator is well-known, see e.g. [16, thm.5.8] or (26) below; that the separable Hölder–Zygmund classes  $\lambda_\alpha(\mathbb{T})$  inherit this property follows from lemma 4 applied to the inequality [20, thm.1.3,III]:

$$\omega_{\widetilde{f}}(\delta) \leq C \left( \int_0^\delta \frac{\omega_f(t)}{t} dt + \delta \int_\delta^\pi \frac{\omega_f(t)}{t^2} dt \right), \quad (26)$$

for a constant  $C$  not depending on  $f$ . Alternatively, one may use here Privalov's theorem [10, thm.6]. Since taking the logarithm is continuous  $\lambda_\alpha^{+,*}(\mathbb{T}) \rightarrow \lambda_\alpha(\mathbb{T})$  and taking the exponential is also continuous on  $\lambda_\alpha(\mathbb{T})$ , gathering things up yields the desired continuity  $\rho \mapsto w_\rho$ .

Finally, when  $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$ , so does  $1/\rho$  and the lemma follows from the identity  $\Omega_\rho^{-1} = \Omega_{1/\rho}$ .  $\blacksquare$

We are now in position to establish the genericity result we are looking for. For both statements and proofs, it is more convenient to handle  $G = f \vee \psi$  as a whole.

**Proposition 3** *The set  $\mathcal{U} \subset \lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$  of pairs  $(G, M)$  such that  $\gamma_N(G, M) \neq 0$ , and also such that the Hankel operator with symbol  $Gw_{\gamma_N \vee (\gamma_N + M)}^{-1}$  has simple  $N+1$ -st singular values, is open and dense.*

*Proof:* set

$$\mathcal{F} := \{G \in \lambda_\alpha(\mathbb{T}), G|_K \in (H^\infty + R_N)|_K\}.$$

We first show that  $\mathcal{F}$  is meagre in  $\lambda_\alpha(\mathbb{T})$ . Indeed, for  $n \in \mathbb{N}$ , let

$$\mathcal{F}_n := \{G \in \mathcal{F}, G|_K = h|_K \text{ where } h \in H^\infty + R_N \text{ satisfies } \|h\|_{L^\infty(\mathbb{T})} \leq n\}.$$

Taking into account that the  $\lambda_\alpha$  topology is finer than the  $L^\infty$  one, it follows from lemma 1 and weak-\* compactness of balls in  $L^\infty(\mathbb{T})$  that  $\mathcal{F}_n$  is closed in  $\lambda_\alpha(\mathbb{T})$  for each  $n$ . Moreover  $\mathcal{F}_n$  has empty interior, because if  $G \in \mathcal{F}_n$  and  $r_{N+1}$  is a rational function having  $N+1$  poles in  $\mathbb{D}$  none of which is a pole of  $G$ , then  $G + \epsilon r_{N+1}$  cannot belong to  $\mathcal{F}_n$  when  $\epsilon > 0$  because  $(G + \epsilon r_{N+1})|_K$  does not even belong to  $(H^\infty + R_N)|_K$ . Appealing to the Baire category theorem,  $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  is meagre in  $\lambda_\alpha(\mathbb{T})$  as announced.

Next, for fixed  $G \in \lambda_\alpha(\mathbb{T}) \setminus \mathcal{F}$ , we define a map

$$\begin{aligned} \Upsilon_G : \lambda_\alpha^{+,0}(\mathbb{T} \setminus K) &\rightarrow \lambda_\alpha^{+,0}(\mathbb{T} \setminus K) \\ M &\mapsto \frac{M}{\gamma_N(G, M)}. \end{aligned}$$

*We claim that  $\Upsilon_G$  is a homeomorphism.*

To prove that it is bijective, we construct the inverse using proposition 2, namely  $\Upsilon_G^{-1}(Y) = \gamma(Y)Y$  where  $\gamma(Y)$  is the unique positive real number such that

$$\sigma_{N+1}(\Gamma_{Gw_{\gamma(Y)}^{-1}w_{1 \vee (1+Y)}^{-1}}) = 1.$$

The continuity of  $\Upsilon_G$  follows from that of  $\gamma_N$  given by proposition 1. The continuity of  $\Upsilon_G^{-1}$  reduces to that of  $\gamma(Y)$  defined above, which can be seen as follows. It is readily checked that  $\gamma = \gamma(Y)$  is the unique solution to

$$\gamma = \gamma_N(G, \gamma Y).$$

Fix  $Y_1 \in \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$  and let  $\lambda_1, \lambda_2 > 0$  be such that  $\lambda_1 < \gamma(Y_1) < \lambda_2$ . Since  $\gamma_N(G, \gamma Y)$  decreases with  $\gamma$ , we necessarily get that

$$\lambda_1 - \gamma_N(G, \lambda_1 Y_1) < 0, \quad \lambda_2 - \gamma_N(G, \lambda_2 Y_1) > 0.$$

By continuity of  $\gamma_N$ , the same holds true for all  $Y_2$  close enough to  $Y_1$  in  $\lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$  implying that also  $\lambda_1 < \gamma(Y_2) < \lambda_2$ .

*This establishes the claim.*

By theorem 7, (ii), and lemma 4, the subset  $\mathcal{O}_{N+1} \subset \lambda_\alpha(\mathbb{T})$  of those  $F$  for which  $\Gamma_F$  has simple  $N+1$ -st singular values is open and dense. Consequently, by lemma 5, for fixed  $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$ , the subset  $\mathcal{O}_\rho = \Omega_\rho^{-1}\mathcal{O}_{N+1} \subset \lambda_\alpha(\mathbb{T})$  of those  $G$  for which  $\Gamma_{Gw_\rho}$  has simple

$N + 1$ -st singular values is also open and dense. Let  $Q$  be a countable dense subset<sup>4</sup> of  $\lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$ . Define a countable subset of  $\lambda_{\alpha}^{+,*}(\mathbb{T})$  by:

$$\mathcal{Q} := \{(1 \vee (1 + \Upsilon_G^{-1}(q)))^{-1}, q \in Q\}.$$

Finally, let

$$\mathcal{G} := \cap_{\rho \in \mathcal{Q}} \mathcal{O}_{\rho} \cap (\lambda_{\alpha}(\mathbb{T}) \setminus \mathcal{F}).$$

By the Baire category theorem,  $\mathcal{G}$  is dense in  $\lambda_{\alpha}(\mathbb{T})$  hence the fibered product:

$$\mathcal{P} := \{(G, M); G \in \mathcal{G}, M \in \Upsilon_G^{-1}(Q)\}$$

is also dense in  $\lambda_{\alpha}(\mathbb{T}) \times \lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$  because  $Q$  is dense in  $\lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$  by definition and  $\Upsilon_G$  is a homeomorphism. Moreover, for each pair  $(G, M) \in \mathcal{P}$ , the Hankel operator with symbol

$$G w_{\gamma_N(G, M) \vee (\gamma_N(G, M) + M)}^{-1} = G \gamma_N(G, M)^{-1} w_{(1 \vee (1 + \Upsilon_G(M)))^{-1}}$$

has simple  $N + 1$ -st singular values. Since  $\mathcal{P} \subset \mathcal{U}$ , it only remains for us to show that  $\mathcal{U}$  is open, and this is easily done. Indeed, in view of proposition 1, the map  $\gamma_N$  is *a fortiori* continuous  $\lambda_{\alpha}(\mathbb{T}) \times \lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K) \rightarrow \mathbb{R}$  thus the condition  $\gamma_N(G, M) > 0$  defines an open set. The condition of non-multiplicity of the Hankel singular values does the same because it behaves continuously in  $\lambda_{\alpha}^{+,*}(\mathbb{T})$  with respect to the symbol:

$$G w_{\gamma_N(G, M) \vee (\gamma_N(G, M) + M)}^{-1},$$

which in turn is continuous with respect to  $(G, M) \in \lambda_{\alpha}(\mathbb{T}) \times \lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$ . ■

*Remark 5* An analogous of proposition 3 holds for  $\beta_N = \beta_N(G, M')$  and the Hankel operator with symbol  $G w_{\beta_N \vee M'}^{-1}$ , with respect to the pairs  $(G, M')$  belonging to

$$\lambda_{\alpha}(\mathbb{T}) \times \{M' \in \lambda_{\alpha}^{+,*}(\mathbb{T} \setminus K), M' = \beta_N \text{ at boundary points of } K\}.$$

## 6 A generic algorithm in Hölder classes

We now describe a convergent procedure to compute the solution to problem 1 in the case where  $(f \vee \psi, M)$  belongs to the open and dense subset  $\mathcal{U}$  of  $\lambda_{\alpha}(\mathbb{T}) \times \lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$  defined in corollary 3.

Write  $G = f \vee \psi$ . We shall assume that we are provided with a sequence  $(r_m)$  of rational functions that converges to  $G$  in  $\lambda_{\alpha}(\mathbb{T})$ . Let  $(\varepsilon_m)$  be a sequence of positive numbers decreasing to 0. An algorithm to solve for problem 1 is obtained by performing, for each  $m$ , the following steps.

1. Find a positive number  $\tilde{\gamma}_m$  such that

$$|1 - \sigma_{N+1}(\Gamma_{r_m w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)}^{-1}})| \leq \varepsilon_m. \quad (27)$$

---

<sup>4</sup>Such a subset exists because  $\lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$  is separable by lemma 4 and every  $M \in \lambda_{\alpha}^{+,0}(\mathbb{T} \setminus K)$  is the trace on  $\mathbb{T} \setminus K$  of some member of  $\lambda_{\alpha}(\mathbb{T})$  obtained for instance by concatenating with the zero function on  $K$ .

2. Compute the best AAK approximant  $v_m$  to  $r_m w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)}^{-1}$  in  $H^\infty + R_N$ .

3. Set

$$h_m = \frac{M}{\tilde{\gamma}_m + M} v_m w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)} + \frac{\tilde{\gamma}_m}{\tilde{\gamma}_m + M} \psi.$$

Then  $h_m$  converges in  $\lambda_\alpha(\mathbb{T} \setminus K)$  to the solution of problem 1 as  $m$  goes to infinity.

Because  $(G, M) \in \mathcal{U}$ , it holds that  $f \notin \mathcal{C}_{M, \psi}^N$ . We show first that  $(\tilde{\gamma}_m)$  is bounded away from 0 and  $\infty$ . It follows from (16) with  $\Phi = r_m$ ,  $\Psi = w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)}^{-1}$ , and  $k = N + 1$  that if  $\tilde{\gamma}_m \rightarrow \infty$  then  $\sigma_{N+1}(\Gamma_{r_m w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)}^{-1}}) \rightarrow 0$ ; the same applies to a subsequence, thereby contradicting (27).

Further  $\tilde{\gamma}_m$  cannot go to 0. Indeed, as a consequence of (27), there exists  $g_m \in H^\infty + R_N$  such that

$$\|r_m w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)}^{-1} - g_m\|_{L^\infty(\mathbb{T})} \leq 1 + \varepsilon_m.$$

Let

$$f_m = g_m w_{\tilde{\gamma}_m \vee (\tilde{\gamma}_m + M)} \in H^\infty + R_N.$$

It then holds that

$$\begin{aligned} |r_m - f_m| &\leq (1 + \varepsilon_m) \tilde{\gamma}_m && \text{a.e. on } K, \\ |r_m - f_m| &\leq (1 + \varepsilon_m) (\tilde{\gamma}_m + M) && \text{a.e. on } \mathbb{T} \setminus K. \end{aligned}$$

Hence  $(f_m)$  is bounded and admits a weak-\* convergent subsequence with limit in  $H^\infty + R_N$ , in view of lemma 1. This limit function is equal to  $f$  a.e. on  $K$  and belongs also to  $\mathcal{B}_{M, \psi}^N$ , a contradiction.

Extracting any subsequence of  $(\tilde{\gamma}_m)$  with limit  $\gamma$ , say, it follows from (27) and lemma 3 that

$$\sigma_{N+1}(\Gamma_{G w_{\gamma \vee (\gamma + M)}^{-1}}) = 1,$$

hence  $\gamma = \gamma_N(G, M)$  by proposition 2. This shows that

$$\lim_{m \rightarrow \infty} |\tilde{\gamma}_m - \gamma_N(G, M)| = 0.$$

That  $v_m$  introduced in step 2 goes to the solution  $v_N$  of the AAK problem (18) with  $\gamma = \gamma_N(G, M)$  is ensured by [29, thm.6] whose hypotheses are verified on  $\mathcal{U}$  by proposition 3 and lemma 5.

Hence,  $h_m$  defined in step 3 converges in  $\lambda_\alpha(\mathbb{T} \setminus K)$  to the solution

$$h_N = \frac{M}{\gamma_N + M} v_N w_{\gamma_N \vee (\gamma_N + M)} + \frac{\gamma_N}{\gamma_N + M} \psi,$$

to problem 1, in view of theorems 3, 4, and proposition 1, point (iii).

Last but not least, each of these steps is constructive. Though we do not discuss here how to obtain the sequence  $(r_m)$ , as this depends much on the way  $G$  is given, we simply note that, if the Fourier coefficients of  $G$  are available, one can use Fejér approximants, as in the proof



of lemma 4, but also Jackson or de La Vallée Poussin trigonometric polynomials. Computing the singular value involved in step 1 and the associated Schmidt pair needed in step 2 (see section 1.3) is straightforward since  $r_m$  is rational. Indeed, if  $r_m = p_\kappa/q_\ell$  with  $p_\kappa \in P_\kappa[z]$ ,  $q_\ell \in P_\ell[z]$ , and for arbitrary  $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$ , we can write:

$$r_m w_\rho = \frac{t_{\ell-1}}{q_\ell} + h$$

where  $h$  belongs to  $H^\infty$  and  $t_{\ell-1} \in P_{\ell-1}[z]$  is easily computed from the values of  $p_m$ ,  $w_\rho$ , and possibly a certain number of their derivatives at the roots of  $q_\ell$ . Now, the outer function  $w_{\gamma \vee (\gamma+M)}$  being numerically available through (9), a solution  $\tilde{\gamma}_m$  to (27) can be computed using a dichotomy procedure.

An analogous algorithm works to solve for problem 2.

## 7 Conclusion

We have given constructive solutions to a pair of extremal problems in  $L^\infty$  that can be regarded as band-limited versions of the Adamjan–Arov–Krein approximation problems. Such approximation problems have recently found a new lease of life in applications to robust identification and model validation. The idea is that they can be used to design nearly optimal algorithms for system modelling from given experimental data. These approximation problems have applications in the solution of related interpolation questions [5, 28], which can be used, for example, to test whether measured experimental data (either frequency response measurements or input/output measurements) are consistent with certain a priori assumptions on the system model.

Finally, we mention as an open question that of finding the closest to analytic extension with fixed real part: this would be useful for the numerical solution of Dirichlet–Neumann problems from incomplete data, which might typically be specified on the arc of the unit circle. We note that there are possible applications of these methods in physical problems such as fault detection.

## References

- [1] V.M. Adamjan, D.Z. Arov, and M.G. Krein. Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur–Takagi problem. *Math. USSR Sbornik*, 15:31–73, 1971.
- [2] L. Aizenberg. *Carleman’s formulas in complex analysis*. Kluwer Academic Publishers, 1993.
- [3] L. Baratchart and J. Leblond. Identification harmonique et trace des classes de hardy sur un arc de cercle. In *Actes du Colloque en l’honneur du 60<sup>e</sup> anniversaire du professeur Jean Céa, Sophia–Antipolis, avril 1992*, pages 17–29. Cépaduès–Editions, 1993.

- [4] L. Baratchart and J. Leblond. Hardy approximation to  $L^p$  functions on subsets of the circle with  $1 \leq p < \infty$ . *Constructive Approximation*, 14:41–56, 1998.
- [5] L. Baratchart, J. Leblond, and J.R. Partington. Constrained  $H^p$  interpolation on subsets of the circle. In *Conference in honor of M. Livšic, Beer-Sheva, June 1997*. Birkhäuser. Submitted.
- [6] L. Baratchart, J. Leblond, and J.R. Partington. Hardy approximation to  $L^\infty$  functions on subsets of the circle. *Constructive Approximation*, 12:423–436, 1996.
- [7] L. Baratchart, J. Leblond, J.R. Partington, and N. Torkhani. Robust identification in the disc algebra from band-limited data. *IEEE Trans. on Aut. Control*, 42:1318–1325, 1997.
- [8] L. Baratchart, J. Leblond, and N. Torkhani. Best bounded  $H^2$  extension of partial frequency data. In A. Isidori, S. Bittanti, E. Mosca, A. De Luca, M.D. Di Benedetto, and G. Oriolo, editors, *Proc. of the 3d European Control Conference, Roma*, volume 2, pages 1330–1335, September 1995.
- [9] L. Baratchart and F. Seyfert. An  $L^p$  analog to the AAK theory. In preparation, 1997.
- [10] H.G.W. Begehr. *Complex analytic methods for partial differential equations*. World Scientific, 1994.
- [11] G. Birkhoff and S. Maclean. *A survey of modern algebra*. Macmillan, New York, 1941.
- [12] G. Chen and J. Zhou. *Boundary element methods*. Computational mathematics and applications. Academic Press, 1992.
- [13] I. Cioranescu. *Geometry of Banach spaces, duality mappings and nonlinear problems*. Kluwer Academic Publishers, 1990.
- [14] D. Colton and R. Kress. *Integral equation methods in scattering theory*. Wiley-Interscience, 1983.
- [15] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*. Springer-Verlag, 1992.
- [16] P.L. Duren. *Theory of  $H^p$  spaces*. Academic Press, 1970.
- [17] B. Francis. *A course in  $H^\infty$  control theory*. Lectures notes in control and information sciences. Springer-Verlag, 1987.
- [18] P.A. Fuhrmann. *A polynomial approach to linear algebra*. Springer, 1996.
- [19] D.J.H. Garling. *A course in Galois theory*. Cambridge University Press, 1986.
- [20] J.B. Garnett. *Bounded analytic functions*. Academic Press, 1981.

- [21] K. Glover. All optimal Hankel-norm approximants of linear multivariable systems and their  $L_\infty$  error bounds. *Int. J. Control*, 39:1115–93, 1984.
- [22] E. Hayashi, L.N. Trefethen, and M.H. Gutknecht. The CF Table. *Constructive Approximation*, 6:195–223, 1990.
- [23] J.W. Helton and O. Merino. *Classical control using  $H^\infty$  methods*. Draft, 1994.
- [24] K. Hoffman. *Banach spaces of analytic functions*. Dover, 1988.
- [25] M.M. Lavrentiev. *Some improperly posed problems of mathematical physics*, volume 11 of *tracts in Natural Philosophy*. Springer, 1967.
- [26] M. Papadimitrakakis. Continuity of the operator of best uniform approximation by bounded analytic functions. *Bull. London Math. Soc.*, 25:44–48, 1993.
- [27] J.R. Partington. *An introduction to Hankel operators*. Cambridge University Press, 1988.
- [28] J.R. Partington. *Interpolation, identification and sampling*. Oxford University Press, 1997.
- [29] V.V. Peller. Hankel operators and continuity properties of the operators of best approximation. *Leningrad Math. J.*, 2(1):139–160, 1991.
- [30] V.V. Peller. Boundedness properties of the operators of best approximation by analytic and meromorphic functions. *Ark. Math.*, 30(2):331–343, 1992.
- [31] V.V. Peller and S.V. Khrushchev. Hankel operators, best approximations, and stationary Gaussian processes. *Russian Math. Surveys*, 37(1):61–144, 1982.
- [32] B. Porat. *Digital processing of random signals: theory and methods*. Prentice-Hall, 1994.
- [33] S.C. Power. *Hankel operators on Hilbert space*. Pitman, Boston, 1982.
- [34] P.A. Regalia. *Adaptive IIR Filtering in Signal Processing and Control*. Dekker, 1995.
- [35] W. Rudin. *Real and complex analysis*. Mc Graw Hill, 1982.
- [36] N.J. Young. *An introduction to Hilbert space*. Cambridge University Press, 1988.
- [37] A. Zygmund. *Trigonometric series*. Cambridge University Press, 1968.



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